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On Some Previous Results for the Drazin Inverse of Block Matrices

Jelena Višnjić[a]

[a] *University of Niš, Faculty of Medicine*
Bul. Dr Zorana Djindjića 81, 18000 Niš, Republic of Serbia

Abstract. This short paper is motivated by the paper of Bu et al. [C. Bu, C. Feng, P. Dong, A note on computational formulas for the Drazin inverse of certain block matrices, J. Appl. Math. Comput.(38) (2012) 631–640], where the authors gave additive formula for Drazin inverse for matrices under new conditions, and two representations under some specific conditions. Here is shown that the additive formula is not valid for all matrices which satisfy given conditions. Also, here is proved that the representations which were given in mentioned paper do not extend the results given by Hartwig et al. [R. Hartwig, X. Li, Y. Wei, Representations for the Drazin inverse of a 2×2 block matrix, SIAM J.Matrix. Anal. Appl. (27)(2006) 757-771], in fact they are equivalent.

1. Introduction

Let $A \in \mathbb{C}^{n\times n}$. By $\mathrm{rank}(A)$ we denote the rank of matrix A. The smallest nonnegative integer k such that $\mathrm{rank}(A^{k+1}) = \mathrm{rank}(A^k)$, denoted by $\mathrm{ind}(A)$, is called the index of A. If $\mathrm{ind}(A) = k$, there exists the unique matrix $A^d \in \mathbb{C}^{n\times n}$, which satisfies the relations:

$$A^{k+1}A^d = A^k,\ A^dAA^d = A^d,\ AA^d = A^dA.$$

The matrix A^d is called the Drazin inverse of A [1]. Clearly, $\mathrm{ind}(A) = 0$ if and only if A is nonsingular, and $A^d = A^{-1}$. In this paper we use notation $A^\pi = I - AA^d$. Also, we agree that $A^0 = I$ and $\sum_{i=1}^{k-j} x_i = 0$, for $k \le j$, where $i, j, k \in \mathbb{N}$.

In 1958 Drazin (see [6]) posed the problem of finding the formula for $(P + Q)^d$ in terms of P, Q, P^d, Q^d, where $P, Q \in \mathbb{C}^{n\times n}$. According to current literature, this problem is still the open one. Anyway, many papers offered a formula for $(P + Q)^d$ with some side conditions for the matrices P and Q (see [5–7, 9, 15]). In [12], the authors derived the formula for $(P + Q)^d$ under condition $P(P + Q)Q = 0$. The case when $(P + Q)P(P + Q) = 0$, $QPQ^2 = 0$ was studied in [2]. In [3], the authors gave the formula for $(P + Q)^d$, which is valid when $(P + Q)P(P + Q)P = 0$, $(P + Q)P(P + Q)Q(P + Q) = 0$, $QPQ^3 = 0$. Here we also give a formula for $(P + Q)^d$ under mentioned conditions, from which we get directly that a formula for $(P + Q)^d$ from [2] is

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Email address: jelena.visnjic@medfak.ni.ac.rs (Jelena Višnjić)

not valid for all matrices which satisfy conditions $(P+Q)P(P+Q)=0$, $QPQ^2=0$. As a corollary we get the correct formula which is valid for all matrices which satisfy mentioned conditions. As an illustration, we offer an example which shows that formula from [2] is not valid for all matrices which satisfy conditions $(P+Q)P(P+Q)=0$, $QPQ^2=0$.

Let M be a complex block matrix of a form:

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}, \tag{1}$$

where A and D are the square matrices, not necessarily of the same size. The problem of expressing M^d in terms of A^d and D^d, with arbitrary blocks A, B, C and D, was posed in 1979 by Campbell and Meyer[4]. This problem still remains open. However, there is a sizable literature on this subject, and there are many representations for M^d, under specific conditions for blocks of M. Some conditions, under which the formula for M^d is known, concern the generalized Schur complement of matrix M, which is defined by $S = D - CA^dB$. Some of them are as follows:

(i) $CA^\pi = 0$, $A^\pi B = 0$ and S is nonsingular or zero (see [11, 14]);

(ii) $CA^\pi B = 0$, $AA^\pi B = 0$ and S is nonsingular or zero (or $CA^\pi B = 0$, $CA^\pi A = 0$ and S is nonsingular or zero)(see [8]);

(iii) $CA^\pi B = 0$, $A^2A^\pi B = 0$, $CAA^\pi B = 0$ and S is nonsingular (or $CA^\pi B = 0$, $CA^\pi A^2 = 0$, $CA^\pi AB = 0$ and S is nonsingular)(see [9]);

(iv) $BCA^\pi B = 0$, $A^2A^\pi B = 0$, $CAA^\pi B = 0$ and $S = 0$ (or $CA^\pi BC = 0$, $CA^\pi A^2 = 0$, $CA^\pi AB = 0$ and $S = 0$) (see [9]);

(v) $CA^\pi BC = 0$, $AA^\pi BC = 0$ and $S = 0$ (or $BCA^\pi B = 0$, $BCA^\pi A = 0$, and $S = 0$) (see [15]);

(vi) $ABCA^\pi A = 0$, $ABCA^\pi B = 0$ and $S = 0$ (or $AA^\pi BCA = 0$, $CA^\pi BCA = 0$ and $S = 0$) (see [13]).

In [2] the authors offered the formula for M^d, considering the case when S is nonsingular, in order to extend representations under conditions (ii) from previous list, which are given in [8]. Here is shown that these conditions are actually equivalent.

2. Auxiliary Lemmas

In this section some needed lemmas are given.

Lemma 2.1. [1] *Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{n\times m}$. Then* $(AB)^d = A((BA)^2)^dB$.

Lemma 2.2. [10] *Let M be matrix of a form*

$$M = \begin{bmatrix} A & B \\ 0 & D \end{bmatrix},$$

where A and D are square matrices such that $\operatorname{ind}(A) = k$, $\operatorname{ind}(D) = l$. *Then* $\max\{k, l\} \le \operatorname{ind}(M) \le k + l$ *and*

$$M^d = \begin{bmatrix} A^d & X \\ 0 & D^d \end{bmatrix},$$

where

$$X = \sum_{i=0}^{l-1} (A^d)^{i+2} B D^i D^\pi + \sum_{i=0}^{k-1} A^\pi A^i B (D^d)^{i+2} - A^d B D^d.$$

Lemma 2.3. [15] *Let* $P, Q \in \mathbb{C}^{n\times n}$ *be such that* $\mathrm{ind}(P) = r$, $\mathrm{ind}(Q) = s$. *If* $PQP = 0$ *and* $PQ^2 = 0$, *then*

$$(P+Q)^d = Y_1 + Y_2 + \left(Y_1(P^d)^2 + (Q^d)^2 Y_2 - Q^d(P^d)^2 - (Q^d)^2P^d\right)PQ,$$

where

$$Y_1 = \sum_{i=0}^{s-1} Q^\pi Q^i (P^d)^{i+1},\ Y_2 = \sum_{i=0}^{r-1} (Q^d)^{i+1} P^i P^\pi. \tag{1}$$

3. Results

Now, we will present our additive formula under conditions $(P+Q)P(P+Q)P = 0$, $(P+Q)P(P+Q)Q(P+Q) = 0$ and $QPQ^3 = 0$, from which we will get the correct formula under conditions $(P+Q)P(P+Q) = 0$ and $QPQ^2 = 0$.

Theorem 3.1. *Let* $P, Q \in \mathbb{C}^{n\times n}$. *If* $(P+Q)P(P+Q)P = 0$, $(P+Q)P(P+Q)Q(P+Q) = 0$ *and* $QPQ^3 = 0$ *then*

$$(P+Q)^d = (P+Q)^2((Q(P+Q))^d)^3(P+Q)^3, \tag{1}$$

where

$$\begin{aligned}((Q(P+Q))^d)^3 &= X_1((QP)^d)^2 + (Q^d)^4X_2 - \sum_{i=1}^{2}(Q^d)^{2i}((QP)^d)^{3-i}\\ &+\left(X_1((QP)^d)^4 + (Q^d)^8X_2 - \sum_{i=1}^{4}(Q^d)^{2i}((QP)^d)^{5-i}\right)QPQ^2,\end{aligned}$$

and

$$X_1 = \sum_{i=0}^{t-1} Q^\pi Q^{2i}((QP)^d)^{i+1},\ X_2 = \sum_{i=0}^{t-1}(Q^d)^{2i+2}(QP)^i(QP)^\pi, \tag{2}$$

for $t = \max\{\mathrm{ind}(QP), \mathrm{ind}(Q^2)\}$.

Proof. Using Lemma 2.1, we have that

$$\begin{aligned}(P+Q)^d &= \begin{bmatrix} I & P\end{bmatrix}\left(\left(\begin{bmatrix} Q \\ I\end{bmatrix}\begin{bmatrix} I & P\end{bmatrix}\right)^2\right)^d\begin{bmatrix} Q \\ I\end{bmatrix}\\ &= \begin{bmatrix} I & P\end{bmatrix}\left(\begin{bmatrix} Q & QP \\ I & P\end{bmatrix}^2\right)^d\begin{bmatrix} Q \\ I\end{bmatrix}\\ &= \begin{bmatrix} I & P\end{bmatrix}\begin{bmatrix} Q(P+Q) & Q(P+Q)P \\ P+Q & (P+Q)P\end{bmatrix}^d\begin{bmatrix} Q \\ I\end{bmatrix}.\end{aligned}$$

Denote by

$$\begin{aligned} M &= \begin{bmatrix} Q(P+Q) & Q(P+Q)P \\ P+Q & (P+Q)P\end{bmatrix},\\ F &= \begin{bmatrix} Q(P+Q) & Q(P+Q)P \\ 0 & (P+Q)P\end{bmatrix},\\ G &= \begin{bmatrix} 0 & 0 \\ P+Q & 0\end{bmatrix}.\end{aligned}$$

Clearly, $M = F + G$. Furthermore, $G^2 = 0$ and $G^d = 0$, $G^\pi = I$ and

$$(P+Q)^d = \begin{bmatrix} I & P \end{bmatrix} M^d \begin{bmatrix} Q \\ I \end{bmatrix}. \tag{3}$$

Evidently, $G^2 = 0$, $G^d = 0$ and $G^\pi = I$. Since $(P+Q)P(P+Q)Q(P+Q) = 0$, we have $FGF = 0$. Hence, matrices F and G satisfy the conditions of Lemma 2.3 and we get

$$M^d = F^d + G(F^d)^2 + (F^d)^2G + G(F^d)^3G. \tag{4}$$

By Lemma 2.2, we get

$$F^d = \begin{bmatrix} (Q(P+Q))^d & \Sigma \\ 0 & ((P+Q)P)^d \end{bmatrix}.$$

From $(P+Q)P(P+Q)P = 0$ we get $((P+Q)P)^d = 0$ and $((P+Q)P)^\pi = I$. Therefore,

$$\begin{aligned}\Sigma &= \left((Q(P+Q))^d\right)^2 Q(P+Q)P + \left((Q(P+Q))^d\right)^3 Q(P+Q)P(P+Q)P \\ &= (Q(P+Q))^dP\end{aligned}$$

Hence,

$$F^d = \begin{bmatrix} (Q(P+Q))^d & (Q(P+Q))^dP \\ 0 & 0 \end{bmatrix}.$$

After some computation, for $k = 1, 2, 3$ we get

$$(F^d)^k = \begin{bmatrix} ((Q(P+Q))^d)^k & ((Q(P+Q))^d)^kP \\ 0 & 0 \end{bmatrix}. \tag{5}$$

After applying (5) into (4) we have

$$M^d = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}, \tag{6}$$

where

$$\begin{aligned}
M_{11} &= (Q(P+Q))^d + \left((Q(P+Q))^d\right)^2 P(P+Q), \\
M_{12} &= (Q(P+Q))^dP, \\
M_{21} &= (P+Q)\left((Q(P+Q))^d\right)^2\left(I + (Q(P+Q))^dP(P+Q)\right), \\
M_{22} &= (P+Q)\left((Q(P+Q))^d\right)^2 P.
\end{aligned}$$

Now, after applying (6) into (3), we get

$$(P+Q)^d = M_{11}Q + PM_{21}Q + M_{12} + PM_{22} = (P+Q)^2((Q(P+Q))^d)^3(P+Q)^3.$$

Hence, (1) holds. Now, it remains to find $((Q(P+Q))^d)^3$. To find $(Q(P+Q))^d = (QP+Q^2)^d$, notice that matrices QP and Q^2 satisfy the conditions of Lemma 2.3. Actually, if we denote by $R = QP$ and $T = Q^2$, since $QPQ^3 = 0$ we have $RTR = 0$ and $RT^2 = 0$. Therefore

$$\begin{aligned}(Q(P+Q))^d \quad &= \quad X_1 + X_2 \\ &\quad +(X_1((QP)^d)^2 + (Q^d)^4X_2 - (Q^d)^2((QP)^d)^2 - (Q^d)^4(QP)^d)QPQ^2,\end{aligned}$$

where X_1 and X_2 are defined by (2).
Notice that $QPQ^d = 0$, $(QP)^d Q^d = 0$, $(QP)^d Q^\pi = (QP)^d$, $(QP)^\pi Q^d = Q^d$ and $QPQ^2 X_1 = 0$. Using these equalities, after some computation we get

$$\begin{aligned}((Q(P+Q))^d)^2 &= X_1(QP)^d + (Q^d)^2 X_2 - (Q^d)^2(QP)^d \\ &\quad + \left(X_1((QP)^d)^3 + (Q^d)^6 X_2 - \sum_{i=1}^{3}(Q^d)^{2i}((QP)^d)^{4-i}\right)QPQ^2,\end{aligned}$$

and also

$$\begin{aligned}((Q(P+Q))^d)^3 &= X_1((QP)^d)^2 + (Q^d)^4 X_2 - \sum_{i=1}^{2}(Q^d)^{2i}((QP)^d)^{3-i} \\ &\quad + \left(X_1((QP)^d)^4 + (Q^d)^8 X_2 - \sum_{i=1}^{4}(Q^d)^{2i}((QP)^d)^{5-i}\right)QPQ^2.\end{aligned}$$

The proof is complete. □

As direct corollary of Theorem 3.1 we get following additive formula.

Corollary 3.2. *Let $P, Q \in \mathbb{C}^{n\times n}$. If $(P+Q)P(P+Q) = 0$ and $QPQ^2 = 0$ then*

$$(P+Q)^d = (P+Q)^2((Q(P+Q))^d)^2(P+Q),$$

where

$$((Q(P+Q))^d)^2 = X_1(QP)^d + (Q^d)^2 X_2 - (Q^d)^2(QP)^d,$$

and X_1, X_2 are defined by (2).

Remark 3.3. *In* [2, Theorem 3.1] *the authors studied conditions from Corollary* 3.2 *and obtained the formula*

$$(P+Q)^d = (P+Q)^2((Q^d)^2 X_2 - (Q^d)^2(QP)^d)(P+Q). \tag{7}$$

Notice that in Corollary 3.2 *we have additional element*

$$(P+Q)^2 X_1(QP)^d(P+Q).$$

Since this element doesn't have to be equal to zero, a formula from [2] *is not correct for all matrices which satisfy conditions $(P+Q)P(P+Q) = 0$ and $QPQ^2 = 0$.*

In next example we consider matrices which satisfy conditions $(P+Q)P(P+Q) = 0$ and $QPQ^2 = 0$ and for which is $(P+Q)^2 X_1(QP)^d(P+Q) \neq 0$.

Example 3.4. *Let P and Q be matrices:*

$$P = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}, \quad Q = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Notice that

$$P + Q = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$$

is idempotent matrix, and therefore $(P+Q)^d = P+Q$.

It can be checked easily that $Q^2 = 0$ and $(P+Q)P(P+Q) = 0$. Hence the conditions of Corollary 3.2 are satisfied. Also, we have that

$$(P+Q)^2 X_1 (QP)^d (P+Q) = (P+Q)^2((QP)^d)^2(P+Q) = P+Q \neq 0.$$

Moreover, $(P+Q)^d = (P+Q)^2 X_1 (QP)^d (P+Q) = (P+Q)$. But, if we apply the formula (7) from [2] we get that $(P+Q)^d = 0$.

At the end of this paper we give the following remark, where we prove that the representations for the Drazin inverse of block matrix M which were given in [2] are equivalent to some representations given in [8].

Remark 3.5. *Consider the block matrix M of a form* (1), *such that $S = D - CA^dB$ is nonsingular. In* [2, Theorem 3.2, Theorem 3.3] *the authors studied following cases:*

(i) $AA^\pi BC = 0$, $AA^\pi BD = 0$, $CA^\pi BC = 0$ *and* $CA^\pi BD = 0$;

(ii) $BCA^\pi A = 0$, $DCA^\pi A = 0$, $BCA^\pi B = 0$ *and* $DCA^\pi B = 0$.

If conditions (i) hold, then

$$AA^\pi BS = AA^\pi BD - AA^\pi BCA^dB = 0,$$
$$CA^\pi BS = CA^\pi BD - CA^\pi BCA^dB = 0.$$

Therefore $AA^\pi B = AA^\pi BSS^{-1} = 0$ and $CA^\pi B = CA^\pi BSS^{-1} = 0$. Hence conditions (i) are equivalent to conditions $AA^\pi B = 0$, $CA^\pi B = 0$ from [8, Theorem 3.1]. *Similarly, if (ii) holds we get $SCA^\pi A = 0$ and $SCA^\pi B = 0$, so conditions (ii) are equivalent to conditions $CA^\pi A = 0$, $CA^\pi B = 0$ from* [8, Corollary 3.2].

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